problem session to section 7.2

Section 7.2

8 p201 $\mathbb{R}^*$ $\mathbb{R}^* \ni 2$ is of infinite order $2, 2^2, 2^3, 2^4 \dots$

The only non-identity element of finite order is $-1 \in \mathbb{R}^*$

order is 2

$-1, 1, -1, 1, \dots$

14 p201 False

$\mathbb{Z}_2 \times \mathbb{Z}_2$ - this of order 4 contains no elements of order 4

16b p201 $|a^k|$ if $|a| = n$

Answer $|a^k| = \dfrac{n}{g.c.d(n,k)}$ For a proof, use Th 7.9

17b p202 Solutions to $ax = b$ and $xa = b$ may be different.

$a^{-1}ax = a^{-1}b$

$x = a^{-1}b$ $x = ba^{-1}$

The problem asks $a, b \in G$ such that $a^{-1}b \neq ba^{-1}$

Let $G = S_3$ $a^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$$a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

**18 p202** $G = \{a_1, a_2, \dots a_n\}$ $(|G| = n)$, $G$ is abelian

Let $x = a_1 \dots a_n$ Wanted: $x^2 = e$ - identity

The list $\{a_1^{-1}, a_2^{-1}, \dots, a_n^{-1}\}$ is also a list of all (distinct) elements of the group.

Note that $a^{-1} = b^{-1}$ implies that $a = b$

Thus $x = a_1 \dots a_n = a_1^{-1} \dots a_n^{-1}$ because both are products of all distinct elements of $G$.

Since $G$ is abelian, $a_1^{-1} \dots a_n^{-1} = (a_1 \dots a_n)^{-1}$

$$\left\{ \begin{array}{l} (ab)^{-1} = \underline{a^{-1} b^{-1}} \\ \quad\; \shortparallel \\ \quad\; \underline{b^{-1} a^{-1}} \end{array} \right.$$

Thus $x = x^{-1}$, equivalently $x^2 = e$.

22 p202 $G = \{e, a, b\}$ Write the operation table for $G$.

| | e | a | b |
|---|---|---|---|
| e | e | a | b |
| a | a | b | e |
| b | b | e | a |

$ab \neq a$ because If $ab = a$ then
$b \neq e$ $a^{-1}ab = a^{-1}a$
$b = e$

Similarly,
$ab \neq b$ $a \neq e$

If $ax = ay$ then $x = y$

If $xa = ya$ then $x = y$

Rem $G$ is a cyclic group of order 3

$e, a, a^2 = b$ $ab = a^3 = e$

$e, b, b^2 = a$

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23 p202 $ab = ca$ implies $b = c$

Wanted: the group is abelian.
$xy = yx$

$y = y$

$ey = ye$ $e = xx^{-1} = x^{-1}x$

$(x^{-1}x)y = y(xx^{-1})$

$x^{-1}(xy) = (yx)x^{-1}$

Let $a = x^{-1}$, $b = xy$, $c = yx$, use the given property

$ab = ca$ implies $\underline{xy = yx}$

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24. p202 $(ab)^2 = a^2b^2$ for all $a, b \in G$.

Wanted: $G$ is abelian
$ab = ba$

$abab = a^2b^2$

$a^{-1}a\underbrace{ba}b\,b^{-1} = \underbrace{a^{-1}a^2}_{a}\,\underbrace{b^2b^{-1}}_{b}$

$\underline{ba = ab}$

$a^{-1}a^2 = (a^{-1}a)a = ea = a$

$b^2b^{-1} = b$

25. p202 $G$ is abelian iff $(ab)^{-1} = a^{-1}b^{-1}$ for every $a, b \in G$.

In any group, $\underline{(ab)^{-1} = b^{-1}a^{-1}}$ because $a(bb^{-1})a^{-1} = e$

If $G$ is abelian, then $(ab)^{-1} = b^{-1}a^{-1} = \underline{a^{-1}b^{-1}}$

Assume $(ab)^{-1} = a^{-1}b^{-1}$ for every $a, b \in G$. Wanted: $ab = ba$

$b^{-1}a^{-1} = a^{-1}b^{-1}$

the inverses of two equal elements are equal

$(b^{-1}a^{-1})^{-1} = ab$ $\qquad (a^{-1})^{-1} = a$

$(a^{-1}b^{-1})^{-1} = ba$

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27, p202 Every nonidentity element of $G$ has order 2

Wanted $G$ is abelian
$ab = ba$

$x^2 = e$ is equivalent to $x = x^{-1}$

- every element of $G$ is equal to its inverse

In particular,

$ab = (ab)^{-1} = b^{-1}a^{-1} = ba$

$b^{-1} = b$

$a^{-1} = a$

30 p202 $\quad a, b \in G$ Prove that $|ab| = |ba|$ $\quad x^n = e \quad n = |x|$

It suffices to prove that $(ab)^n = e$ implies that $(ba)^n = e$ for an integer $n$

$(ab)^n = e$

$\underbrace{ab\, ab \ldots ab}_{n \text{ times}} = e$

$a (ba)^{n-1} b = e$

$(ba)^{n-1} = a^{-1} b^{-1} = (ba)^{-1}$

$(ba)^n = e$

32 p202 $\quad$ If $|G|$ is even then $G$ has an element of order 2.

Pf $\quad$ Every $a \in G$ has $a^{-1} \in G$

Define the sets $A, B \subset G$ as follows:

Pick an element $a \in G$, and put it into $A$, while $a^{-1}$ into $B$ if $a^{-1} \neq a$

such that $a^{-1} \neq a$

Continue untill $G$ is exhausted (we run out of elements $a$ such that $a^{-1} \neq a$.)

Clearly $|A| = |B|$ at every step

If identity $e$ was the only element $x \in G$ with the property $x^{-1} = x$, then $|G| = |A| + |B| + |\{e\}|$ would be odd.

Thus there exists $a \in G$, $a \neq e$ such that $\underline{a^{-1} = a}$.

Note that $a^{-1} = a$ is equivalent to $\underline{a^2 = e}$ which means $\underline{|a| = 2}$

<u>34 p202</u> $\underline{G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2}$